Basic Mathematics

Factorising Expressions

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic competence at factorising simple algebraic expressions.

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1. Factorising Expressions (Introduction)

Expressions such as $(x + 5)(x - 2)$ were met in the package on brackets. There the emphasis was on the expansion of such expressions, which in this case would be $x^2 + 3x - 10$. There are many instances when the *reverse* of this procedure, i.e. factorising, is required. This section begins with some simple examples.

Example 1 Factorise the following expressions.

$$\text{(a) } 7x - x^2, \quad \text{(b) } 2abx + 2ab^2 + 2a^2b.$$

Solution

(a) This is easy since $7x - x^2 = x(7 - x)$.

(b) In this case the largest common factor is $2ab$ so

$$2abx + 2ab^2 + 2a^2b = 2ab(x + b + a).$$

On the next page are some exercises for you to try.

Exercise 1. Factorise each of the following expressions *as far as possible*. (Click on green letters for solutions.)

(a) $x^2 + 3x$ (b) $x^2 - 6x$
(c) $x^2y + y^3 + z^2y$ (d) $2ax^2y - 4ax^2z$
(e) $2a^3b + 5a^2b^2$ (f) $ayx + yx^3 - 2y^2x^2$

Quiz Which of the expressions below is the *full* factorisation of

$$16a - 2a^2\,?$$

(a) $a(16 - 2a)$ (b) $2(8 - 2a)$
(c) $2a(8 - a)$ (d) $2a(4 - 2a)$

Quiz Which of the following is the *full* factorisation of the expression

$$ab^2c - a^2bc^3 + 2abc^2\,?$$

(a) $abc(b - ac^2 + 2c)$ (b) $ab^2(c - ac^3 + ac)2$
(c) $ac(b^2 - abc^2 + 2bc)$ (d) $b^2c(a - abc^2 + ac)$

2. Further Expressions

Each of the previous expressions may be factored in a single operation. Many examples require more than one such operation. On the following page you will find some worked examples of this type.

Example 2 Factorise the expressions below *as far as possible*.

$$\text{(a)} \quad ax + ay + bx + by\,, \qquad \text{(b)} \quad 6ax - 3bx + 2ay - by\,.$$

Solution

(a) Note that a is a factor of the first two terms, and b is a factor of the second two. Thus

$$ax + ay + bx + by = a(x + y) + b(x + y)\,.$$

The expression in this form consists of a sum of two terms, each of which has the common factor $(x + y)$ so it may be further factorised. Thus

$$\begin{aligned} ax + ay + bx + by &= a(x + y) + b(x + y) \\ &= (a + b)(x + y)\,. \end{aligned}$$

(b) Here $3x$ is a factor of the first two terms and y is a factor of the second two. Thus

$$\begin{aligned} 6ax - 3bx + 2ay - by &= 3x(2a-b) + y(2a-b) \\ &= (3x+y)(2a-b), \end{aligned}$$

taking out $(2a-b)$ as a common factor.

Exercise 2. Factorise each of the following *as fully as possible*. (Click on green letters for solution.)

(a) $xb + xc + yb + yc$
(b) $ah - ak + bh - bk$
(c) $hs + ht + ks + kt$
(d) $2mh - 2mk + nh - nk$
(e) $6ax + 2bx + 3ay + by$
(f) $ms + 2mt^2 - ns - 2nt^2$

Quiz Which of the following is the factorisation of the expression

$$2ax - 6ay - bx + 3by\,?$$

(a) $(2a+b)(x+3y)$
(b) $(2a-b)(x-3y)$
(c) $(2a+b)(x-3y)$
(d) $(2a-b)(x+3y)$

3. Quadratic Expressions

A *quadratic* expression is one of the form $ax^2 + bx + c$, with a, b, c being some *numbers*. When faced with a quadratic expression it is often, *but not always*, possible to *factorise it by inspection*. To get some insight into how this is done it is worthwhile looking at how such an expression is formed.

Suppose that a quadratic expression can be factored into two linear terms, say $(x + d)$ and $(x + e)$, where d, e are two *numbers*. Then the quadratic is

$$\begin{aligned}(x + d)(x + e) &= x^2 + xe + xd + de\,,\\ &= x^2 + (e + d)x + de\,,\\ &= x^2 + (d + e)x + de\,.\end{aligned}$$

Notice how it is formed. The coefficient of x is $(d + e)$, which is the *sum* of the two numbers in the linear terms $(x + d)$ and $(x + e)$. The final term, the one *without* an x, is the *product* of those two numbers. This is the information which is used to *factorise by inspection*.

Example 3 Factorise the following expressions.

$$\text{(a) } x^2 + 8x + 7\,, \quad \text{(b) } y^2 + 2y - 15\,.$$

Solution

(a) The only possible factors of 7 are 1 and 7, and these do add up to 8, so

$$x^2 + 8x + 7 = (x + 7)(x + 1)\,.$$

Checking this (see the package on Brackets for FOIL):

$$\begin{aligned}(x + 7)(x + 1) &= \overset{\mathbf{F}}{x^2} + \overset{\mathbf{O}}{x.1} + \overset{\mathbf{I}}{x.7} + \overset{\mathbf{L}}{7.1} \\ &= x^2 + 8x + 7\,.\end{aligned}$$

(b) Here the term independent of x (i.e. the one without an x) is *negative*, so the two numbers must be opposite in sign. The obvious contenders are 3 and -5, or -3 and 5. The first pair can be ruled out as their sum is -2. The second pair sum to $+2$, which is the correct coefficient for x. Thus

$$y^2 + 2y - 15 = (y - 3)(y + 5)\,.$$

Here are some examples for you to try.

Exercise 3. Factorise the following into *linear* factors. (Click on green letters for solution.)

(a) $x^2 + 7x + 10$
(b) $x^2 + 7x + 12$
(c) $y^2 + 11y + 24$
(d) $y^2 - 10y + 24$
(e) $z^2 - 3z - 10$
(f) $a^2 - 8a + 16$

Quiz Which of the following is the factorisation of the expression

$$z^2 - 6z + 8\,?$$

(a) $(z - 1)(z + 8)$
(b) $(z - 1)(z - 8)$
(c) $(z - 2)(z + 4)$
(d) $(z - 2)(z - 4)$

4. Quiz on Factorisation

Begin Quiz Factorise each of the following and choose the solution from the options given.

1. $$2a^2e - 5ae^2 + a^3e^2$$

(a) $ae(2a - 5e + a^2e)$ (b) $a^2e(2a - 5e + ae)$
(c) $ae(2a - 5e^2 + a^2e^2)$ (d) $a^2e(2 - 5e + a^2e^2)$

2. $$6ax - 3bx + 2ay - by$$

(a) $(3x - y)(2a + b)$ (b) $(3x + y)(2a - b)$
(c) $(3x - y)(2a - b)$ (d) $(3x + y)(2a + b)$

3. $$z^2 - 26z + 165$$

(a) $(z + 11)(z + 15)$ (b) $(z - 11)(z - 15)$
(c) $(z - 55)(z - 3)$ (d) $(z + 55)(z - 3)$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) The only common factor of the two terms is x so

$$x^2 + 3x = x(x + 3)\,.$$

Click on green square to return

□

Exercise 1(b) Again the two terms in the expression have only the common factor x, so

$$x^2 - 6x = x(x - 6)\,.$$

Click on green square to return

Exercise 1(c) Here the only common factor is y so

$$x^2y + y^3 + z^2y = y(x^2 + y^2 + z^2)\,.$$

Click on green square to return □

Exercise 1(d) In this case the largest common factor is $2ax^2$, so

$$2ax^2y - 4ax^2z = 2ax^2(y - 2z)\,.$$

Click on green square to return □

Exercise 1(e)

Here the largest common factor is a^2b, so this factorises as

$$2a^3b + 5a^2b^2 = a^2b(2a + 5b)\,.$$

Click on green square to return □

Exercise 1(f) The largest common factor is xy so

$$ayx + yx^3 - 2y^2x^2 = xy(a + x^2 - 2xy)\,.$$

Click on green square to return ☐

Exercise 2(a) We proceed as follows:

$$\begin{aligned} xb + xc + yb + yc &= x(b+c) + y(b+c) \\ &= (x+y)(b+c)\,. \end{aligned}$$

Click on green square to return

□

Exercise 2(b)

$$\begin{aligned} ah - ak + bh - bk &= a(h-k) + b(h-k) \\ &= (a+b)(h-k)\,. \end{aligned}$$

Click on green square to return

□

Exercise 2(c)

$$\begin{aligned} hs + ht + ks + kt &= h(s+t) + k(s+t) \\ &= (h+k)(s+t)\,. \end{aligned}$$

Click on green square to return ☐

Exercise 2(d)

$$\begin{aligned} 2mh - 2mk + nh - nk &= 2m(h-k) + n(h-k) \\ &= (2m+n)(h-k)\,. \end{aligned}$$

Click on green square to return □

Exercise 2(e)

$$\begin{aligned} 6ax + 2bx + 3ay + by &= 2x(3a + b) + y(3a + b) \\ &= (2x + y)(3a + b) \end{aligned}$$

Click on green square to return □

Exercise 2(f)

$$\begin{aligned} ms + 2mt^2 - ns - 2nt^2 &= m(s + 2t^2) - n(s + 2t^2)(s + 2t^2) \\ &= (m - n)(s + 2t^2) \end{aligned}$$

Click on green square to return □

Exercise 3(a)

Since 10 has the factors 5 and 2, and their *sum* is 7,

$$\begin{aligned}(x+5)(x+2) &= x^2+2x+5x+10\\ &= x^2+7x+10\,.\end{aligned}$$

Click on green square to return

Exercise 3(b)

Here there are several ways of factorising 12 but on closer inspection the only factors that work are 4 and 3. This leads to the following

$$\begin{aligned}(x+4)(x+3) &= x^2+3x+4x+12 \\ &= x^2+7x+12\,.\end{aligned}$$

Click on green square to return

Exercise 3(c)

There are several different possible factors for 24 but only one pair, 8 and 3 add up to 11. Thus

$$\begin{aligned}(y+8)(y+3) &= y^2+3y+8y+24\\ &= y^2+11y+24\,.\end{aligned}$$

Click on green square to return □

Exercise 3(d)

There are several different possible factors for 24 but only one pair, 6 and 4 add up to 10. Since the coefficient of y is negative, and the constant term is positive, the required numbers this time are -6 and -4. Thus

$$\begin{aligned}(y-6)(y-4) &= y^2 - 4y - 6y + (-6)(-4) \\ &= y^2 - 10y + 24\,.\end{aligned}$$

Click on green square to return □

Exercise 3(e) The constant term in this case is negative. Since this is the *product* of the numbers required, they must have *opposite* signs, i.e. one is positive and one negative. In that case, the number in front of the x must be the *difference* of these two numbers. On inspection, 5 and 2 have product 10 and difference 3. Since the x term is negative, the larger number must be negative.

$$\begin{aligned}(z-5)(z+2) &= z^2+2z-5z+(-5\times 2)\\ &= z^2-3z-10\,.\end{aligned}$$

Click on green square to return □

Exercise 3(f)

This is an example of a perfect square. These are mentioned in the package on Brackets. The factors of 16 in this case are -4 and -4.

$$\begin{aligned}(a-4)^2 &= (a-4)(a-4)\\ &= a^2-4a-4a+(-4)\times(-4)\\ &= a^2-8a+16\,.\end{aligned}$$

Click on green square to return □

Solutions to Quizzes

Solution to Quiz: Here 2 is a factor of both terms, but so is a, so the *largest common factor* is $2a$. Thus

$$16a - 2a^2 = 2a(8 - a)\,.$$

End Quiz

Solution to Quiz:
The largest common factor in this case is $a \times b \times c = abc$. Thus

$$\begin{aligned} ab^2c - a^2bc^3 + 2abc^2 &= (abc \times b) - (abc \times ac^2) + (abc \times 2c) \\ &= abc(b - ac^2 + 2c) \end{aligned}$$

End Quiz

Solution to Quiz: Noting that $2a$ is a factor of the first two terms and $-b$ is a factor of the second two, we have

$$\begin{aligned} 2ax - 6ay - bx + 3by &= 2a(x - 3y) - b(x - 3y) \\ &= (2a - b)(x - 3y) \end{aligned}$$

End Quiz

Solution to Quiz: Here the two numbers have product 8, so a possible choice is 2 and 4. However their sum in this case is 6, whereas the sum required is -6. Taking the pair to be -2 and -4 will give the same product, $+8$, but with the correct sum. Thus

$$z^2 - 6z + 8 = (z - 4)(z - 2)\,,$$

and this can be checked by expanding the brackets.

End Quiz